

ARE EBERLEIN-GROTHENDIECK SCATTERED SPACES σ -DISCRETE?

ANTONIO AVILÉS AND DAVID GUERRERO SÁNCHEZ

Abstract

A space X is Eberlein-Grothendieck if $X \subset C_p(K)$ for some compact space K . In this paper we address the problem of whether such a space X is σ -discrete whenever it is scattered. We show that if $w(K) \leq \omega_1$ then X is σ -discrete whenever X has height ω_1 and it is locally compact or locally countable. It is also proved that every Lindelöf Čech-complete scattered space is σ -compact.

Keywords: Eberlein-Grothendieck space, locally compact scattered space, locally countable scattered space, meta-Lindelöf space, Čech-complete space

Mathematical subject code: 54C35.

1. INTRODUCTION

In [Ar] Arhangel'skii defined Eberlein-Grothendieck spaces as those homeomorphic to a subspace of $C_p(K)$ for some compact space K . Notice that if X is a subset of a Banach space E with the weak topology then X embeds in $C_p(\mathbf{B}E^*)$ hence X is Eberlein-Grothendieck. The main purpose of this paper is to introduce the following problem:

Problem 1.1. *Are Eberlein-Grothendieck scattered spaces σ -discrete?*

A particular case of this problem was posed in [Hay] where Haydon asked if for every compact K the space $C_p(K, \{0, 1\})$ is σ -discrete whenever it is scattered. This kind of question is related to the following notions introduced by J.E. Jayne, I. Namioka and C.A. Rogers in [JNR]. Given a set X , a metric ρ on X and $\varepsilon > 0$, a family \mathcal{A} of subsets of X is ε -small if $\text{diam}_\rho(A) < \varepsilon$ for every $A \in \mathcal{A}$. A topological space (X, τ) has the property SLD with respect to a metric ρ on X if for every $\varepsilon > 0$ there is a countable cover $\{X_n : n \in \omega\}$ of X such that for each $n \in \omega$ the space X_n admits a τ -open cover which is ε -small. On the other hand a topological space (X, τ) is σ -fragmented by a metric ρ on X if for every $\varepsilon > 0$ there is a countable cover $\{X_n : n \in \omega\}$ of X such that for each $n \in \omega$ and every $Y \subset X_n$ there exists a nonempty

relative τ -open subset U of Y with $\text{diam}_\rho(U) < \varepsilon$. It is clear that if a topological space has SLD with respect to some metric, then it is σ -fragmented as well, but the following is an open question:

Problem 1.2. *Are the properties of σ -fragmentability and SLD equivalent when X is a Banach space endowed with its weak topology and with its norm metric, or when X is of the form $C_p(K)$ endowed with the uniform metric?*

This problem has its origin in the theory of renorming of Banach spaces, as these properties are conjectured as possible internal characterizations of Banach spaces admitting a norm with the Kadets-Klee property. We refer to [MOTV, Section 3.2, p.54] for information about this topic. It is easy to see that if a space with the discrete metric is SLD then it is σ -discrete and if it is σ -fragmented then it is σ -scattered. Thus Problem 1.1 can be viewed as the discrete version of Problem 1.2. Indeed the following question is open as well:

Problem 1.3. *If $C_p(K, \{0, 1\})$ is scattered (respectively σ -discrete), does it imply that $C_p(K)$ is σ -fragmentable (respectively SLD)?*

A version of this problem states the same thing considering the weak topology of $C(K)$ instead of the pointwise convergence topology. The answer is known to be positive when K is scattered (see [Hay] and [Mtz].) Since the restriction of the uniform metric of $C_p(K)$ to $C_p(K, \{0, 1\})$ is discrete, a positive answer to Problems 1.1 and 1.3 combined would give a positive answer to Problem 1.2 in the case of spaces of continuous functions.

We observe that it follows from known results that Problem 1.1 has positive solution when X is compact: Alster proved in [Al] that an Eberlein compact scattered space K is strong Eberlein which implies that K embeds into $\{0, 1\}^\Gamma$ for some Γ and $|\text{supp}(x)| < \omega$ for every $x \in K$. For each $n \in \omega$ we can define $X_n = \{x \in K : |\text{supp}(x)| = n\}$ hence we can write $K = \bigcup X_n$ where each X_n is discrete. In this paper we will prove some generalizations of this fact.

In Section 3, we give some partial positive answers to Problem 1.1. The first one is the following:

Theorem 1.4. *If X is an Eberlein-Grothendieck locally compact scattered space of height lower than $\omega_1 \cdot \omega_1$, then X is σ -discrete.*

In the same section we will prove also our second result which states:

Theorem 1.5. *If X is an Eberlein-Grothendieck locally countable scattered space of cardinality ω_1 , then X is σ -discrete.*

Remember that a transfinite sequence $\{x_\alpha : \alpha < \lambda\}$ of elements of a topological space is right-separated if for every $\mu < \lambda$ there is an open set U such that $U \cap \{x_\alpha : \alpha < \lambda\} = \{x_\alpha : \alpha < \mu\}$. A topological space is scattered if and only if it can be written as a right separated sequence $X = \{x_\alpha : \alpha < \lambda\}$. From this point of view, Theorem 1.5 implies that Problem 1.1 has a positive solution in the first non-trivial case, when $\lambda = \omega_1$.

Corollary 1.6. *If $X = \{x_\alpha : \alpha < \omega_1\} \subset C_p(K)$ is a right-separated ω_1 -sequence, then X is σ -discrete.*

In both of our results mentioned above, X is homeomorphic to some $X' \subset C_p(K)$ where K has weight ω_1 . By [DJP, Theorem 1.2] in this case the space X is hereditarily meta-Lindelöf. This is the hypothesis that we shall actually assume, so that Theorem 1.4 is proved by applying the ideas developed in [Al] to show that every open cover of a hereditarily meta-Lindelöf locally compact scattered space of height at most ω_1 has a point finite clopen refinement, while Theorem 1.5 is proved by showing that hereditarily meta-Lindelöf locally countable scattered spaces are σ -discrete. The latter fact is stated in [HP] without proof, but the argument that they suggest does not seem to be correct. Section 3 ends with a corollary that, at least when K is scattered, the SLD property of $C_p(K)$ can be characterized as a kind of ω_1 - σ -fragmentability.

In Section 4 we deal with another generalization of the Eberlein compact scattered spaces which is the class of Eberlein-Grothendieck Lindelöf Σ scattered spaces. In this case we show that these spaces are σ -discrete as an easy consequence of the results in [Ha] and [Ny]. But for the special subcase of the Eberlein-Grothendieck Lindelöf Čech-complete scattered spaces we actually prove their σ -compactness applying the methods of topological games developed in [Te].

Since the results in Section 3 depend very strongly on the property of hereditary metalindelöfness of the Eberlein-Grothendieck spaces considered in that section, we decided to ask if such property is enough for a scattered topological space to be σ -discrete. In other words, is every hereditarily meta-Lindelöf scattered space σ -discrete? We show in Section 5 that this is not the case in general by constructing a hereditarily meta-lindelöf scattered space that is not σ -discrete. However, the space constructed in section 5 is not even a Hausdorff space, therefore it is not Eberlein-Grothendieck.

In section 6 we pose some of the remaining open problems related to determine whether Eberlein-Grothendieck scattered spaces are σ -discrete. The general question here is still open so we consider several subcases. The most important of these subcases is stated in Problem 6.1 which reads: is every Eberlein-Grothendieck Lindelöf scattered space σ -discrete?

2. NOTATION AND TERMINOLOGY

Unless otherwise stated, every topological space in this article is assumed to be Tychonoff. The topology of X is denoted by $\tau(X)$ and $\tau^*(X)$ is the family of non-empty open subsets of X . For $C \subset X$ the family of all open sets of X that contain C is denoted by $\tau(C, X)$; if $x \in X$ then we write $\tau(x, X)$ instead of $\tau(\{x\}, X)$.

The space of all continuous functions from a space X into a space Y , endowed with the topology inherited from the product space Y^X , is denoted by $C_p(X, Y)$. The space $C_p(X, \mathbb{R})$ will be abbreviated by $C_p(X)$. For every $f \in C_p(X, Y)$, define the dual map $f^* : C_p(Y) \rightarrow C_p(X)$ by $f^*(g) = g \circ f$ for every $g \in C_p(Y)$. For a space $X \subset C_p(Y)$ the diagonal product of the elements of X is the map $\varphi : Y \rightarrow \mathbb{R}^X$ defined by $\varphi(y)(f) = f(y)$ for any $y \in Y$ and $f \in X$.

A space X is Eberlein-Grothendieck if there is a compact space K such that X is homeomorphic to a subspace of $C_p(K)$. A space is scattered if every subspace of it contains an isolated point. Given a family \mathcal{F} of subsets of a space X , a refinement of \mathcal{F} is a family \mathcal{G} such that $\bigcup \mathcal{F} = \bigcup \mathcal{G}$ and for each $G \in \mathcal{G}$ there is $F \in \mathcal{F}$ such that $G \subset F$. A σ -compact (σ -countably compact) space is the countable union of compact (countably compact) spaces. A space is cosmic if it has a countable network.

As in [Bu], a space X is metacompact if every open cover of X has a point finite open refinement. Given a family \mathcal{F} of subsets of a space X , for every point $x \in X$, define $ord(x, \mathcal{F}) = |\{F \in \mathcal{F} : x \in F\}|$. A space X is weakly θ -refinable if every open cover of X has an open refinement $\mathcal{V} = \bigcup_{n \in \omega} \mathcal{V}_n$ such that for each $x \in X$ there is $n \in \mathbb{N}$ such that $1 \leq ord(x, \mathcal{V}_n) < \omega$. The rest of our terminology is standard and can be found in the books [Ar], [Tk2] and [En].

3. HEREDITARILY META-LINDELÖF EBERLEIN-GROTHENDIECK SPACES

The first mandatory step is to pose the problem as one of covering properties; we can do it by recalling that Nyikos established in [Ny,

Theorem 3.4] that a scattered space is σ -discrete if and only if it is hereditarily weakly θ -refinable. It is clear that this result implies that every metrizable scattered space is σ -discrete as it was proved by Telgarsky in [Te, Theorem 12.1] by other methods. Yakovlev showed in [Ya] that every Eberlein compact space is hereditarily σ -metacompact hence hereditarily weakly θ -refinable, so it suffices to invoke [Ar, Corollary III.4.2] to conclude that every Eberlein-Grothendieck pseudocompact scattered space is σ -discrete.

In [DJP, Theorem 1.2] it was proved by Dow, Junnila and Pelant that if K is a compact space of weight ω_1 then $C_p(K)$ is hereditarily meta-Lindelöf. It follows that every Eberlein-Grothendieck space of cardinality ω_1 is hereditarily meta-Lindelöf. Indeed, suppose that $X \subset C_p(K)$ where K is a compact space such that $w(K) > \omega_1 = |X|$. Let $\varphi(p)(f) = f(p)$ for any $p \in K$ and $f \in X$. In [Tk2, Problem 166] it is proved that $\varphi(p) \in C_p(X) \subset \mathbb{R}^{\omega_1}$, for any $p \in K$ and that the map $\varphi : K \rightarrow C_p(X)$ is continuous. The map φ thus defined is called the diagonal product of the elements of X . Let $L = \varphi(K) \subset \mathbb{R}^{\omega_1}$, it is clear that $w(L) = \omega_1$. Besides, the function $\varphi^* : C_p(L) \rightarrow C_p(K)$ defined by $\varphi^*(f) = f \circ \varphi$ embeds $C_p(L)$ into $C_p(K)$ as a closed subspace (see [Tk2, Problem 163]). For every $g \in X$, if $\pi_g : L \rightarrow \mathbb{R}$ is the projection of L onto the factor determined by g , then $g = \varphi^*(\pi_g)$. It follows that the space X embeds into $C_p(L)$ which is hereditarily meta-Lindelöf. We can conclude that every Eberlein-Grothendieck space of cardinality ω_1 is hereditarily meta-Lindelöf.

In this section we will strongly exploit metalindelöfness in Eberlein-Grothendieck spaces combined with a fundamental fact proved by Alster in [Al] which establishes that every locally countable family of compact scattered open subspaces of a space X has a point finite clopen refinement. In order to apply these results to the study of locally compact scattered spaces we will apply some of the ideas introduced in [Sp].

Lemma 3.1. *Every open cover of a hereditarily meta-Lindelöf locally compact scattered space of countable height has a point finite clopen refinement.*

PROOF. Let $\kappa < \omega_1$ be the height of X . For each $\alpha < \kappa$ denote by $X^{(\alpha)}$ the α -th scattering level of the space X . Recalling that a scattered locally compact space is zero dimensional, it is easy to see that every open cover \mathcal{O} of X has a clopen refinement \mathcal{U} such that for every $\alpha < \kappa$ and each $x \in X^{(\alpha)}$ there is $U_x \in \mathcal{U}$ such that $U_x \cap X^{(\alpha)} = \{x\}$ and $U_x \cap X^{(\beta)} = \emptyset$ for every $\beta > \alpha$. We will show that \mathcal{U} has a point finite clopen refinement.

Fix $\alpha < \kappa$. The space $Y_\alpha = \bigcup \{U_x : x \in X^{(\alpha)}\}$ is meta-Lindelöf therefore the cover $\{U_x : x \in X^{(\alpha)}\}$ of Y_α has a point countable refinement \mathcal{U}_α . For each $x \in X^{(\alpha)}$ we can choose a single $U_x^\alpha \in \mathcal{U}_\alpha$ such that $x \in U_x^\alpha \subset U_x$. Moreover, for each $x \in X^{(\alpha)}$ there is a clopen set V_x^α such that $x \in V_x^\alpha \subset U_x^\alpha$. Observe that if $x, y \in X^{(\alpha)}$ and $x \neq y$ then $V_x^\alpha \neq V_y^\alpha$ which implies $U_x^\alpha \neq U_y^\alpha$ because $U_x^\alpha \cap U^{(\alpha)} = \{x\} = V_x^\alpha \cap U^{(\alpha)}$. Therefore, since the family \mathcal{U}_α is point countable so is the family $\{V_x^\alpha : x \in X^{(\alpha)}\}$. We can now apply [Al, Proposition] to find a point finite clopen refinement \mathcal{V}_α of the family $\{V_x^\alpha : x \in X^{(\alpha)}\}$.

Finally, the family $\mathcal{V} = \bigcup_{\alpha \in \kappa} \mathcal{V}_\alpha$ is a point countable clopen refinement of \mathcal{U} . Apply [Al, Proposition] once more to conclude that \mathcal{V} , and hence \mathcal{U} , has a point finite clopen refinement.

Theorem 3.2. *Every open cover of a locally compact scattered hereditarily meta-Lindelöf space of height ω_1 has a point finite clopen refinement.*

PROOF. Let \mathcal{O} be an open cover of X . For each $\alpha < \omega_1$ denote by $X^{(\alpha)}$ the α -th scattering level of X . As in Lemma 3.1 the open cover \mathcal{O} of X has a clopen refinement \mathcal{U}' such that for every $\alpha < \omega_1$ and each $x \in X^{(\alpha)}$ there is $U_x \in \mathcal{U}'$ such that $U_x \cap X^{(\alpha)} = \{x\}$ and $U_x \cap X^{(\beta)} = \emptyset$ for every $\beta > \alpha$. We will show that \mathcal{U}' has a point finite clopen refinement.

There is a point countable open refinement \mathcal{U} of the cover \mathcal{U}' . We will show that \mathcal{U} has a point finite clopen refinement. Fix $U \in \mathcal{U}$, let $\kappa < \omega_1$ be the height of U . For each $\alpha < \kappa$ and every $x \in U^{(\alpha)}$ there is a clopen $V_x^\alpha \subset U$ such that $V_x^\alpha \cap U^{(\alpha)} = \{x\}$ and $V_x^\alpha \cap U^{(\beta)} = \emptyset$ for every $\beta > \alpha$. Let $\mathcal{V}_\alpha = \{V_x^\alpha : x \in U^{(\alpha)}\}$. The space U is a hereditarily meta-Lindelöf locally compact scattered space of countable height, therefore we can invoke Lemma 3.1 to find a point finite clopen refinement \mathcal{C}_U of the family $\mathcal{V} = \bigcup_{\alpha \in \kappa} \mathcal{V}_\alpha$. It follows that the family $\mathcal{C} = \bigcup_{U \in \mathcal{U}} \mathcal{C}_U$ is a point countable clopen refinement of \mathcal{U} and its elements are compact scattered subspaces of X . By [Al, Proposition] the cover \mathcal{C} , and hence \mathcal{U} has a point finite clopen refinement.

Corollary 3.3. *Every hereditarily meta-Lindelöf locally compact scattered space of height ω_1 is metacompact.*

In the proof of [Ny, Theorem 3.4] Nyikos proved implicitly the following fact.

Fact 1. *Let κ be a limit ordinal and suppose that X is a scattered space of height κ for which every $U \in \tau(X)$ of height less than κ is σ -discrete. The space X is σ -discrete if it is weakly θ -refinable.*

Metacompactness implies weakly θ -refinability, thus we obtain the following consequence.

Theorem 3.4. *Every hereditarily meta-Lindelöf locally compact space X of height lower than $\omega_1 \cdot \omega_1$ is σ -discrete.*

PROOF. Take any $U \in \tau(X)$, let κ be the height of U and suppose that $\omega_1 \leq \kappa < \omega_1 \cdot \omega_1$. Then $\kappa = \omega_1 \cdot \alpha + \beta$ for some countable ordinals α, β . It is easy to see that Theorem 3.2 implies that the set $\bigcup \{U^{(\gamma)} : \gamma < \omega_1 \cdot \alpha\}$ is σ -discrete. It follows that U is σ -discrete and we can apply the metacompactness of X and Fact 1 to conclude that X is σ -discrete.

Theorem 3.5. *Every scattered hereditarily meta-Lindelöf locally countable space is σ -discrete.*

PROOF. Let \mathcal{C} be a point countable cover of the space X such that every $C \in \mathcal{C}$ is countable. For each $x \in X$, define $U_0^x = \bigcup \{C \in \mathcal{C} : x \in C\}$. Suppose that for every $x \in X$ we have defined the countable open set $U_n^x \in \tau(x, X)$. Let $U_{n+1}^x = \bigcup \{U_0^y : y \in U_n^x\}$ and $V_x = \bigcup \{U_n^x : n \in \omega\}$. Notice that by this construction, for every $x \in X$ and every $C \in \mathcal{C}$ if $C \cap V_x \neq \emptyset$ then $C \subset V_x$.

Now if $x \in V_y$ then, there is $n \in \omega$ such that $x \in U_n^y$ which implies that $U_0^x \subset U_{n+1}^y$. It follows that $V_x \subset V_y$. On the other hand, by the construction, $x \in U_n^y$ also implies that $y \in U_n^x$ and thus $V_y \subset V_x$. We can conclude that if $V_y \cap V_x \neq \emptyset$ then $V_y = V_x$.

We have shown that the cover $\{V_x : x \in X\}$ induces a partition $\mathcal{P} = \{P_\alpha : \alpha \in I\}$ of the space X by countable open subsets of X . We can write $P_\alpha = \{x_n^\alpha : n \in \omega\}$ and define $D_n = \{x_n^\alpha : \alpha \in I\}$ for every $n \in \omega$. It is clear that $X = \bigcup \{D_n : n \in \omega\}$ and that each D_n is discrete because P_α isolates x_n^α in D_n .

We have already observed that Eberlein-Grothendieck spaces of cardinality ω_1 are hereditarily meta-Lindelöf. This observation allows us to obtain the following corollaries.

Corollary 3.6. *Every Eberlein-Grothendieck locally compact scattered space of cardinality ω_1 and height lower than $\omega_1 \cdot \omega_1$ is σ -discrete.*

Corollary 3.7. *Every Eberlein-Grothendieck locally countable scattered space of cardinality ω_1 is σ -discrete.*

Corollary 3.8. *Every Eberlein-Grothendieck right-separated transfinite sequence $X = \{x_\alpha : \alpha < \lambda\}$ with $\lambda < \omega_1 \cdot \omega_1$ is σ -discrete.*

If we add separability as a hypothesis in Corollary 3.7, then we have the following direct consequence of [Sp, Lemma 2.14].

Corollary 3.9. *Suppose that K is a compact space of weight ω_1 , if $X \subset C_p(K)$ is locally countable separable and scattered then X is countable.*

As we already mentioned in the Introduction, when K is a compact scattered space, the σ -fragmentability of $C_p(K)$ is equivalent to $C_p(K, \{0, 1\})$ being σ -scattered, and analogously the SLD property of $C_p(K)$ is equivalent to $C_p(K, \{0, 1\})$ being σ -discrete, cf. [Mtz]. The following definition is a slight variation of [Fa, Definition 5.1.1]

Definition 3.10. *Let X be a space with a metric ρ . An increasing family $\{U_\beta : \beta < \alpha\} \subset \tau^*(X)$ that covers X and $\text{diam}_\rho(U_\gamma \setminus \bigcup_{\beta < \gamma} U_\beta) < \varepsilon$ is called an ε -open partitioning of length α of the space X .*

Corollary 3.11. *If K is a scattered compact space of weight ω_1 then $C_p(K)$ is SLD if and only if for every $\varepsilon > 0$ there exists a family $\{X_n : n \in \omega\}$ that covers $C_p(K, \{0, 1\})$ and for each $n \in \omega$ the set X_n has an ε -open partitioning of length ω_1 .*

PROOF. Take $\varepsilon < 1$, there exists a family $\{X_n : n \in \omega\}$ that covers $C_p(K)$ and for each $n \in \omega$ the set X_n has a ε -open partitioning $\{U_\alpha^n : \alpha < \omega_1\}$. For every $\alpha < \omega_1$ we have $|U_\alpha^n \cap C_p(K, \{0, 1\})| \leq 1$ which implies that for each $n \in \omega$, the space $X_n \cap C_p(K, \{0, 1\})$ is a scattered locally countable space thus σ -discrete by Corollary 3.8 and hence $C_p(K, \{0, 1\})$ is σ -discrete as well. Apply [Mtz, Theorem 6] to conclude that $C_p(K)$ is SLD.

Assume that $C_p(K)$ is SLD and take $\varepsilon > 0$ and a countable cover $\{X_n : n \in \omega\}$ such that for each $n \in \omega$ the space X_n has an ε -small open cover. For every $n \in \omega$ we have $l(X_n) \leq w(C_u(K)) = w(K) = \omega_1$, therefore X_n has an ε -small open cover \mathcal{V}_n of cardinality ω_1 . Let $\mathcal{V}_n = \{V_\beta^n : \beta < \omega_1\}$ and define $U_\beta^n = \bigcup_{\alpha \leq \beta} V_\alpha^n$. The family $\{U_\beta^n : \beta < \omega_1\}$ is an ε -open partitioning of length ω_1 .

4. TOPOLOGICAL GAMES ON EBERLEIN-GROTHENDIECK SPACES

In [Ha] R.W. Hansell proved, among other things, that for every compact K that is a continuous image of a Valdivia compact, the

space $C_p(K)$ is hereditarily weakly θ -refinable. Thus if K is a continuous image of a Valdivia compact then each scattered subspace of $C_p(K)$ is σ -discrete. This implies in particular that if X is an Eberlein-Grothendieck Lindelöf Σ scattered space then it is σ -discrete. Indeed, suppose that $X \subset C_p(Q)$ for some compact Q and let φ be the diagonal product of the elements of X . If $K = \varphi(Q)$ then the dual map φ^* embeds the space $C_p(K)$ into $C_p(Q)$ as a closed subspace that contains X (see [Tk2, Problem 163]). It follows that $C_p(K)$ contains a homeomorphic copy of X that is a Lindelöf Σ subspace of $C_p(K)$ which separates the points of K . This implies that $C_p(K)$ is Lindelöf Σ by [Ar, Corollary IV.2.10] and therefore K is a Gul'ko (and hence Valdivia) compact space; thus X is hereditarily weakly θ -refinable and consequently σ -discrete. As a consequence, every Eberlein-Grothendieck K -analytic or cosmic scattered space is σ -discrete.

Since every Lindelöf Čech-complete space X is Lindelöf Σ (see for example [Tk1, Theorem 1]) we already know that if X is scattered then it is σ -discrete. However, we can say more in this case; in this section we will apply the methods developed in [Te] to prove that Lindelöf Čech-complete scattered spaces are in fact σ -compact.

Definition 4.1. *On a Tychonoff space Y , consider a family \mathcal{C} of subsets of Y . We define the game $\mathcal{G}(\mathcal{C}, Y)$ of two players I and II who take turns in the following way: at the move number n , Player I chooses $C_n \in \mathcal{C}$ and Player II chooses a set $U_n \in \tau(C_n, Y)$. The game ends after the n -th move of each player has been made for every $n \in \omega$ and Player I wins if $X = \bigcup \{U_n : n \in \omega\}$; otherwise the winner is Player II.*

Definition 4.2. *A strategy t for the first player in the game $\mathcal{G}(\mathcal{C}, Y)$ on a space X is defined inductively in the following way. First the set $t(\emptyset) = F_0 \in \mathcal{C}$ is chosen. An open set $U_0 \in \tau(X)$ is legal if $F_0 \subset U_0$. For every legal set U_0 the set $t(U_0) = F_1 \in \mathcal{C}$ has to be defined. Let us assume that legal sequences (U_0, \dots, U_i) and sets $t(U_0, \dots, U_i)$ have been defined for each $i \leq n$. The sequence (U_0, \dots, U_{n+1}) is legal if so is the sequence (U_0, \dots, U_i) for each $i \leq n$ and $F_{n+1} = t(U_0, \dots, U_n) \subset U_{n+1}$. A strategy t for Player I is winning if it ensures victory for I in every play it is used.*

Recall that given a space X and $Y \subset X$, the set Y has countable character in X if there exists a countable family $\mathcal{U} \subset \tau(Y, X)$ such that for every $V \in \tau(Y, X)$ there exists $U \in \mathcal{U}$ such that $V \subset U$. A space X is of countable type if every compact subset of X is contained in a compact space of countable character in X . Notice that in a space

of countable type X not necessarily every compact subset of X has countable character in X , therefore the following proposition does not follow from the results in [Te], however we will apply the ideas in [Te] to prove it.

Proposition 4.3. *If a space X has countable type and \mathcal{K} is the family of its compact subspaces, then the first player has a winning strategy for the game $\mathcal{G}(\mathcal{K}, X)$ if and only if X is σ -compact.*

PROOF. It suffices to prove necessity. Suppose that the space X has countable type and that the first player has a winning strategy for the game $\mathcal{G}(\mathcal{K}, X)$. Let s be a winning strategy for the first player in the game $\mathcal{G}(\mathcal{K}, X)$. For every $F \in \mathcal{K}$ there is $K(F) \in \mathcal{K}$ such that $F \in K(F)$ and $\chi(K(F)) \leq \omega$ hence, for each $F \in \mathcal{K}$ it is possible to find a countable family $\mathcal{U}_F = \{U_n^F : n < \omega\} \subset \tau(K(F), X)$ such that $\bigcap \mathcal{U}_F = K(F)$. For the compact set $F_0 = s(\emptyset)$ define $A_0 = \{K(F_0)\}$ and for every $n \in \mathbb{N}$ define $A_n = \{K(s(U_{l_0}^{F_0}, \dots, U_{l_{n-1}}^{F_{n-1}})) : l_0, \dots, l_{n-1} < \omega, F_i \in A_i \text{ for each } i < n \text{ and } ((F_0, U_{l_0}^{F_0}), \dots, (F_{n-1}, U_{l_{n-1}}^{F_{n-1}})) \text{ is an initial segment of a match of the game } \mathcal{G}(\mathcal{K}, X) \text{ in which the first player applies the strategy } s\}\}$. Observe that $|A_n| \leq \omega$ for every $n \in \omega$. Indeed, $|A_0| \leq \omega$. If we assume that $|A_n| \leq \omega$ then we have that $|A_{n+1}| \leq |A_n| \cdot |\mathcal{U}_{F_n}| \leq \omega^2 \leq \omega$. Therefore $|\bigcup \{A_n : n \in \omega\}| \leq \omega$.

For every $n \in \omega$ define $B_n = \bigcup A_n$ and $B = \bigcup B_n$. We will show that $X = B$. Suppose that $y \in X \setminus B$; this implies $y \neq K(F_0)$ thus there is $U_0 \in \mathcal{U}_{F_0}$ such that $y \notin U_0$. Let $F_1 = s(U_0)$. The set $K(F_1) \in A_1$ therefore $y \notin K(F_1)$ and there is $U_1 \in \mathcal{U}_{F_1}$ such that $y \notin U_1$. Let $F_2 = s(U_1, U_2)$. Suppose F_k, U_{k-1} have been defined by this procedure in such a way that $y \notin U_j$ with $j = 1, \dots, k-1$, it is then possible to find $U_k \in \mathcal{U}_{F_k}$ such that $y \notin U_k$. By the definition of $\{U_n : n \in \omega\}$ we have that $\mathcal{P} = \{(F_n, U_n) : n \in \omega\}$ is a match of $\mathcal{G}(\mathcal{K}, X)$ in which the first player applies s , but $y \notin U_n$ for every $n \in \omega$. This contradiction shows that X is σ -compact.

Let \mathcal{S} be the family of singletons of a space Lindelöf scattered space X . In [Te, Theorem 9.3] Telgarsky proved that the first player has a winning strategy for the game $\mathcal{G}(\mathcal{S}, X)$. Thus we have the following corollary.

Corollary 4.4. *Every Lindelöf scattered space of countable type is σ -compact.*

The most important class of countable type spaces are the Čech-complete spaces, so we obtain the following theorem.

Theorem 4.5. *Every Lindelöf Čech-complete scattered space is σ -compact.*

Let \mathcal{S} be the family of singletons of a compact space K . In [Te] it is proved, among other things, that the first player has a winning strategy for the game $\mathcal{G}(\mathcal{S}, K)$ if and only if K is scattered. We can apply this fact in the context of Eberlein-Grothendieck spaces to obtain the following corollary.

Corollary 4.6. *Let \mathcal{S} be the family of all singletons of an Eberlein-Grothendieck Čech-complete space X . If the first player has a winning strategy for the game $\mathcal{G}(\mathcal{S}, X)$ then X is σ -discrete.*

PROOF. The space X being Čech-complete has countable type. Besides by [Te, Corollary 2.2], the first player also has a winning strategy for the game $\mathcal{G}(\mathcal{K}, X)$, where \mathcal{K} is the family of compact subsets of X . Apply Theorem 4.3 to conclude that X is σ -compact. We can write $X = \bigcup_{n \in \omega} K_n$ where each K_n is Eberlein compact. It is easy to see that for each $n \in \omega$ the first player has a winning strategy for the game $\mathcal{G}(\mathcal{S}_n, K_n)$, where \mathcal{S}_n is the family of singletons of K_n implying that each K_n is scattered and consequently σ -discrete and so is X .

5. EXAMPLE

In Section 3 we applied the hereditary metalindelöfness of certain scattered spaces to prove they are σ -discrete. It is not very clear that this property implies σ -discreteness of scattered Tychonoff spaces. This is not true for general spaces as we can deduce from the following example.

Example 5.1. *There exists a scattered space of class T_1 which is hereditarily meta-Lindelöf that is not σ -discrete.*

PROOF. We will define a T_1 hereditarily meta-Lindelöf topology on the ordinal $\omega_1 \cdot \omega_1$. First we will define some auxiliary sets. Fix an ordinal $\gamma < \omega_1$ and define $V_{(\alpha, \gamma)}$ as follows:

- $V_{(\alpha, \gamma)} = \emptyset$ if $\alpha < \gamma$.
- $V_{(\gamma, \gamma)} = [\omega_1 \cdot \gamma, \omega_1 \cdot (\gamma + 1))$.
- $V_{(\gamma + \beta, \gamma)} = [\omega_1 \cdot \gamma + \beta, \omega_1 \cdot (\gamma + 1))$ for $\beta < \omega_1$.

Define $V_\alpha = \bigcup_{\gamma < \omega_1} V_{(\alpha, \gamma)}$.

Observe that the family $\{V_\alpha : \alpha < \omega_1\}$ thus defined is point countable. Indeed, let $\xi \in \omega_1 \cdot \omega_1$. There are countable ordinals ς, ϱ such that $\xi = \omega_1 \cdot \varsigma + \varrho$. It is clear that $\xi \notin V_\alpha$ for $\alpha > (\varsigma + \varrho) + 1$.

For every $\lambda < \omega_1 \cdot \omega_1$ let $U_\lambda = [0, \lambda)$ and $W_\alpha^\lambda = V_\alpha \cap U_\gamma$. It is easy to see that the family $\mathcal{W} = \{W_\alpha^\lambda : \alpha < \omega_1, \lambda < \omega_1 \cdot \omega_1\}$ is a base for a topology τ on $\omega_1 \cdot \omega_1$. Note that by this construction $U_\lambda \in \tau$ for each $\lambda \in \omega_1 \cdot \omega_1$; hence the space $(\omega_1 \cdot \omega_1, \tau)$ with the order of $\omega_1 \cdot \omega_1$ is right-separated, and therefore this space is scattered.

To verify that the space $(\omega_1 \cdot \omega_1, \tau)$ is hereditarily meta-Lindelöf it suffices to show that for any family $\mathcal{C} \subset \mathcal{W}$ there is a point countable family $\mathcal{O} \subset \tau$ such that $\bigcup \mathcal{C} = \bigcup \mathcal{O}$.

Take a family of basic open sets $\mathcal{C} = \{W_\alpha^\lambda : (\alpha, \lambda) \in I\} \subset \mathcal{W}$. For each $\alpha < \omega_1$ define $C_\alpha = \bigcup \mathcal{C} \cap V_\alpha$ and $J_\alpha = \{\lambda \in \omega_1 \cdot \omega_1 : W_\alpha^\lambda \in \mathcal{C}\}$. Notice that the family $\{C_\alpha : \alpha < \omega_1\}$ is point countable, therefore it will be enough to show a point countable family \mathcal{O}_α such that $C_\alpha = \bigcup \mathcal{O}_\alpha$ for each $\alpha < \omega_1$.

There are three possible mutually exclusive cases:

- (1) If there is $\xi \in J_\alpha$ such that $W_\alpha^\lambda \subset W_\alpha^\xi$ for every $\lambda \in J_\alpha$ then define $\mathcal{O}_\alpha = \{W_\alpha^\xi\}$
- (2) If J_α is bounded in V_α and $cf(J_\alpha) = \omega$, then let J'_α be a countable cofinal subset of J_α and $\mathcal{O}_\alpha = \{W_\alpha^\lambda : \lambda \in J'_\alpha\}$.
- (3) If the cofinality of the set J_α is ω_1 then it is possible to find an increasing ω_1 -sequence of ordinals $\{\lambda_\eta : \eta < \omega_1\} \subset (\omega_1 \cdot \beta, \omega_1 \cdot (\beta + 1))$, with $\beta + 1 \leq \alpha + 1$, that is cofinal in J_α . For each $\eta < \omega_1$ there is $\delta(\eta) < \omega_1$ such that $\lambda_\eta = \omega_1 \cdot \beta + \delta(\eta)$. Let $\mathcal{O}_\eta = \{W_\varsigma^{\lambda_{(\eta+1)}} : \beta + \delta(\eta) \leq \varsigma < \beta + \delta(\eta + 1)\}$. Define $\mathcal{O}_\alpha = \bigcup_{\eta < \omega_1} \mathcal{O}_\eta$. To see that in this case the family \mathcal{O}_α is point

countable, take $\rho \in \bigcup \mathcal{O}_\alpha$. Since each \mathcal{O}_η is countable, it suffices to show that the set $\{\eta < \omega_1 : \rho \in \bigcup \mathcal{O}_\eta\}$ is countable. We can find countable ordinals ζ and ξ such that $\rho = \omega_1 \cdot \zeta + \xi$. Observe that by the definition of each $\delta(\eta)$ the family $\{\delta(\eta) : \eta < \omega_1\}$ is increasing and cofinal in ω_1 . Thus there is $\eta < \omega_1$ such that $\zeta + \xi + 1 < \delta(\eta)$. It is clear that $\rho \notin \mathcal{O}_{\eta'}$ for every $\eta' > \eta$.

To see that X is not σ -discrete, suppose that $X = \bigcup_{n \in \omega} D_n$. For each $\alpha < \omega_1$ define $D_n^\alpha = D_n \cap [\omega_1 \cdot \alpha, \omega_1 \cdot (\alpha + 1))$. For every $\alpha \in \omega_1$ we can find $\Phi(\alpha) \in \omega$ such that $|D_{\Phi(\alpha)}^\alpha| = \omega_1$. Thus, there is $m \in \omega$ such that $|\{\alpha < \omega_1 : |D_m^\alpha| = \omega_1\}| = |\Phi^{-1}(m)| = \omega_1$. It is possible to find ordinals $\beta \in \omega_1$ y $\gamma \in D_m$ such that $|\{\alpha \in \beta : |D_m^\alpha| = \omega_1\}| = \omega$ and $\omega_1 \cdot \beta < \gamma \in D_m$.

If $\gamma \in W_\delta^\xi \in \tau$ then for every $\alpha < \beta$ such that $|D_m^\alpha| = \omega_1$ there exist $\lambda < \omega_1$ such that $W_\delta^\xi \cap [\omega_1 \cdot \alpha, \omega_1 \cdot (\alpha + 1)) = [\omega_1 \cdot \alpha + \lambda, \omega_1 \cdot (\alpha + 1))$. Since $|D_m^\alpha| = \omega_1$, it follows that $D_m \cap [\omega_1 \cdot \alpha + \lambda, \omega_1 \cdot (\alpha + 1)) \neq \emptyset$. This shows that the set D_m is not a discrete subspace of $(\omega_1 \cdot \omega_1, \tau)$.

Example 6.1 shows that hereditary metalindelöfness alone is not enough to guarantee σ -discreteness of a scattered topological spaces in general. Unfortunately, this example does not apply to the context of Eberlein-Grothendieck spaces because it is not even a Hausdorff space. Indeed, any two uncountable ordinals cannot be separated by disjoint open subsets. Hence a natural question arises: is every Eberlein-Grothendieck hereditarily metalindelöf scattered space σ -discrete? (see Problem 6.4).

6. OPEN PROBLEMS

It is already known that Eberlein scattered spaces are σ -discrete, the next evident step would be to consider the case of Eberlein-Grothendieck Lindelöf spaces. A positive answer would follow if $C_p(K)$ was hereditarily weakly θ -refinable whenever K is a compact subspace of $C_p(L)$ for a scattered Lindelöf space L .

Problem 6.1. *Is every Eberlein-Grothendieck Lindelöf scattered space σ -discrete?*

Problem 6.2. *Suppose that L is a Lindelöf scattered space and K is a compact subspace of $C_p(L)$. Is $C_p(K)$ hereditarily weakly θ refinable?*

In Section 4 we showed that Problem 1.1 has a positive partial answer for the case of Lindelöf Čech-complete scattered spaces. What if we remove the hypothesis that the space is Lindelöf?

Problem 6.3. *Is every Eberlein-Grothendieck Čech-complete scattered space σ -discrete?*

As noticed in Section 5, if we consider spaces that are not Tychonoff, hereditarily metalindelöfness does not necessarily imply σ -discreteness of scattered spaces in general. Does it in Eberlein-Grothendieck spaces?

Problem 6.4. *Is every Eberlein-Grothendieck hereditarily meta-Lindelöf scattered space σ -discrete?*

Problem 6.3 would be a positive result, assuming MA, if the following question had an affirmative answer.

Problem 6.5. *Let X be an Eberlein-Grothendieck hereditarily meta-Lindelöf scattered space of height and cardinality equal to ω_1 . For every*

point $x \in X$ there is an open set U_x that isolates x in its scattering level. There is a point countable open refinement \mathcal{V} of the cover $\{U_x : x \in X\}$. Let V_x be the union of all the elements of \mathcal{V} that contain x . Define the partially ordered set $P = \{p \subset X : p \text{ is finite and } V_x \cap p = \{x\} \text{ for every } x \in p\}$ and $q < p$ if $p \subset q$. Has P the countable chain condition?

Take an Eberlein-Grothendieck right-separated transfinite sequence $X = \{x_\alpha : \alpha < \lambda\}$. In Section 3 we showed that X is hereditarily metalindelöf for $\lambda < \omega_2$. Moreover we proved that hereditarily metalindelöfness implies σ -discreteness of X for $\lambda < \omega_1 \cdot \omega_1$. It is not yet clear if hereditarily metalindelöfness implies σ -discreteness of X for $\lambda = \omega_1 \cdot \omega_1$.

Problem 6.6. Suppose that $\omega_1 \cdot \omega_1 \leq \lambda < \omega_2$ and $X = \{x_\alpha : \alpha < \lambda\}$ is an Eberlein-Grothendieck right-separated transfinite sequence. Is X σ -discrete?

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DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE MURCIA, 30100 MURCIA, SPAIN

E-mail address: avileslo@um.es,david.guerrero@um.es